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# ON THE CHARACTERISTICS AND SOLUTIONS OF THE ONE-DIMENSIONAL NON-STATIONARY SEEPAGE EQUATION†

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The characteristic equations for the one-dimensional non-stationary seepage equation are presented. It is shown that any exact solution of the equation of the characteristics has a constant arbitrariness (of no more than three arbitrary constants). This arbitrariness is used to construct, generally speaking, an approximate solution of the initial boundary-value problem. 

## 1. FORMULATION OF THE PROBLEM

For the non-stationary seepage equation in the one-dimensional case

$$p_{xx} = mp^{-\gamma/(\gamma+1)}p_t/(k\gamma) \tag{1.1}$$

($m$ is the porosity of the medium, $k$ is the seepage coefficient, $\gamma$ is the polytropic exponent, $p$ is the pressure in the medium, $t$ is the time, $x$ is the spatial variable and differentiation with respect to the corresponding variable is denoted by subscripts) which, in a certain modified time scale, has the form

$$p_t = p_x^2/\gamma + pp_{xx} \tag{1.2}$$

an initial boundary-value problem has been set up with the following initial and boundary conditions

$$p(x, 0) = 0, \quad p(0, t) = F(t), \quad F(0) = 0 \tag{1.3}$$

This problem is considered below.

Choosing the time $t = t(x, p)$ as the dependent variable, we write Eq. (1.2) in the form

$$t_p^2 - t_x^2 t_p/\gamma + p(t_{xx}t_p^2 - 2t_x t_{xp} t_p + t_x^2 t_{pp}) = 0 \tag{1.4}$$

If $x$ is chosen as the dependent variable, then, for the function $x = x(p, t)$ we obtain the equation

$$x_t x_p^2 + x_p/\gamma - px_{pp} = 0 \tag{1.5}$$

Equations (1.2), (1.4) and (1.5) have a singularity accompanying the higher derivatives and are characterized by a finite rate of propagation of perturbations [2].

Exact solutions, having a constant arbitrariness, have been obtained for such equations by different methods in a number of papers (see [3–7], for example) but the question remains as to why the exact solutions of Eqs (1.2), (1.4) and (1.5) only have a constant arbitrariness.

A study of the characteristics of Eq. (1.1) enables us to answer this question.

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## 2. INVESTIGATION OF THE CHARACTERISTICS

In Eq. (1.4), we change to the new independent variables

$$p-\varphi(x) = \xi, \quad x = \eta \tag{2.1}$$

It is obvious that the Jacobian of the transformation is not equal to zero.

We now consider under what condition $\xi$ = const is a characteristic of Eq. (1.4).

In the new variables, Eq. (1.4) has the form

$$\begin{aligned} &t_\xi^2 - t_\xi(t_\eta - \varphi_x t_\xi)^2/\gamma + (\varphi+\xi)[(t_{\eta\eta} - 2\varphi_x t_{\xi\eta} + \varphi_x^2 t_{\xi\xi} - \varphi_{xx}t_\xi)t_\xi^2 - \\ &- 2t_\xi(t_\eta - \varphi_x t_\xi)(t_{\xi\eta} - \varphi_x t_{\xi\xi}) + (t_\eta - \varphi_x t_\xi)^2 t_{\xi\xi}] = 0 \end{aligned} \tag{2.2}$$

We collect the coefficients of the derived derivative $t_{\xi\xi}$ and require that their sum should be equal to zero. The line $\xi$ = const is then a characteristic [8]. We obtain

$$\varphi_x^2 t_\xi^2 + 2\varphi_x t_\xi(t_\eta - \varphi_x t_\xi) + (t_\eta - \varphi_x t_\xi)^2 = 0$$

We now write the relation which must be satisfied on the characteristic in order that Eq. (2.2) should have a solution. We obtain the ordinary differential equation (ODE)

$$1/t_\xi - \varphi_x^2/\gamma - (\varphi+\xi)\varphi_{xx} = 0 \tag{2.3}$$

which must satisfy the characteristics of Eq. (2.2). The solutions of the Eq. (2.3) depend on the parameter $\xi$ = const.

Equation (2.3) has the singular solution

$$\varphi(x) = \pm(x\sqrt{\gamma/t_\xi}) + a(\xi), \quad a(\xi) = \text{const} \tag{2.4}$$

If $\varphi_x \neq 0$, then, on putting $\varphi_x(x) = y(\varphi)$, for the function $y^2(\varphi) = q$ we obtain the first-order ODE

$$[(\varphi+\xi)/2]q_\varphi = 1/t_\xi - q/\gamma$$

It follows from this that

$$q^{1/2} = \varphi_x = u, \quad u = \pm\sqrt{\gamma\left(\frac{1}{t_\xi} - \frac{c(\xi)}{(\varphi+\xi)^{2/\gamma}}\right)} \tag{2.5}$$

Here, $c(\xi)$ is an arbitrary constant.

We now write the solution of (2.5) with separable variables, expressing the function $\varphi(x)$ in terms of $u$ and obtain

$$A(\xi)\int\frac{dv}{(1\pm v^2)^{1+\gamma/2}} = x + b(\xi), \quad v = u\sqrt{\frac{|t_\xi|}{\gamma}} \tag{2.6}$$

where $A(\xi)$ and $b(\xi)$ are certain constants.

If $\gamma = 2(n-1)$, where $n$ is an integer, then, on carrying out the integration in (2.6), we obtain [9] (also, see [4])

$$x = -b(\xi) + A(\xi)W$$

$$W = \frac{(2n-3)!!}{2^n(n-1)!}z_\pm + \frac{v}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)(1\pm v^2)^{n-k}} \tag{2.7}$$

$$z_- = \ln\frac{1+v}{1-v}, \quad z_+ = 2\operatorname{arctg} v$$

If a minus (plus) sign is under the integral sign in formula (2.6), $z_-(z_+)$ is chosen.
When $\gamma = 2n - 1$ [9]

$$W = \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1}\binom{n-1}{k}\frac{(\pm 1)^k v^{2k+1}}{(1 \pm v^2)^{k+1/2}}$$

The plus (minus) signs in the numerator and denominator of this expression are chosen to be the same and correspond to the signs in formula (2.6).
If $\gamma$ is any number, then [9]

$$x = -b(\xi) + vF(\gamma/2 + 1, 1/2; 3/2; -v^2) \tag{2.8}$$

Here $F(\alpha, \beta; \nu; x)$ is hypergeometric function.
Expressions (2.7) and (2.8) are, in fact, solutions of Eq. (1.5) for a fixed time (see relations (2.1) and (2.5)). Consequently, for any $t$, the solution of Eq. (1.5) in the case of arbitrary $\psi$ will have the form

$$x(p,t) = b(t) + A(t)B^{1/2}\left[1 + \frac{1}{2}\sum_{i=1}^{\infty} \frac{(\gamma/2+1)(\gamma/2+2)\dots(\gamma/2+i)}{(1/2+i)i!}B^i\right]$$
$$B = 1 - a(t)p^{-2/\gamma} \tag{2.9}$$

Substituting expression (2.9) into Eq. (1.5), we obtain a system of first-order ODEs for determining the functions $a(t), A(t)$ and $b(t)$, and, on solving this system, we will have the solution of Eq. (1.5) which depends on three arbitrary constants.
We now consider the case when $\gamma = -1$. In this case, Eq. (2.3) has the solution

$$\varphi(x) = -\xi + a(\xi)\cos x + b(\xi)\sin x, \quad a(\xi)^2 + b(\xi)^2 = -1/t_\xi, \quad t_\xi < 0$$

$$\varphi(x) = -\xi + c(\xi)\operatorname{ch} x + d(\xi)\operatorname{sh} x, \quad c(\xi)^2 + d(\xi)^2 = 1/t_\xi, \quad t_\xi > 0$$

It follows from this that

$$p(x,t) = r(t) + a(t)\cos x + b(t)\sin x, \quad t_\xi < 0 \tag{2.10}$$

$$p(x,t) = f(t) + c(t)\operatorname{ch} x + d(t)\operatorname{sh} x, \quad t_\xi > 0 \tag{2.11}$$

Substituting expression (2.10) into Eq. (1.2), we obtain (differentiation with respect to time is denoted by a dot)

$$\dot{r} + \dot{a}\cos x + \dot{b}\sin x = -(b\cos x - a\sin x)^2 - (r+h)h, \quad h = b\sin x + a\cos x$$

This expression will be an identity if

$$\dot{r} = -(a^2 + b^2), \quad \dot{a} = -ra, \quad \dot{b} = -rb \tag{2.12}$$

We shall assume (the general case) that $a(t) \neq 0$ and $b(t) \neq 0$. Then,

$$r^2 + \nu = a^2 + b^2, \quad \nu = \text{const}$$

Further, suppose $\alpha = \sqrt{|\nu|}$. Then [5]

$$p(x,t) = \alpha\operatorname{tg}[\alpha(\beta - t)] + (\lambda\cos x + \mu\sin x)/\cos[\alpha(\beta - t)], \quad \lambda^2 + \mu^2 = \nu > 0$$

$$p(x,t) = \alpha\operatorname{th}[\alpha(\beta + t)] + (\lambda\cos x + \mu\sin x)/\operatorname{ch}[\alpha(\beta + t)], \quad \lambda^2 + \mu^2 = \nu < 0$$

Here, $\alpha$, $\beta$ and $\lambda$ are arbitrary constants.

Similarly, relation (2.11) can be substituted into Eq. (1.2) to obtain a solution which depends on three arbitrary constants.

If, in Eq. (1.4) we change to the variables

$$x - \varphi(p) = \xi, \quad p = \eta \tag{2.13}$$

then $\xi = \text{const}$ will be a characteristic if $t_\eta = 0$.

In this case, it is necessary that the condition

$$\varphi_p^2/t_\xi + \varphi_p/\gamma - p\varphi_{pp} = 0 \tag{2.14}$$

is satisfied on the characteristic.

Let us put $\varphi_p = q$. Then, we shall have an equation for $q$ and, on solving this, we obtain

$$q = \varphi_p = \frac{\sigma p^{1/\gamma}}{\gamma(1 - \varpi p^{1/\gamma})}, \quad \sigma = \text{const}, \quad \varpi = \frac{\sigma}{t_\xi}$$

We shall assume that $\gamma = k/l$, where $l$ and $k$ are integers, $k > l + 1$. Then, on integrating the expression for $q$, we obtain the solution of Eq. (2.14)

$$\varphi(p) = -\frac{l}{k}ap - \frac{l}{k-l}\frac{a}{b^l}p^{(k-l)/k} + cX(\theta)$$

$$a = t_\xi = \text{const}, \quad b = \varpi^{1/l} = \text{const}, \quad c = ab^{ml} = \text{const}, \quad \theta = bp^{1/k}$$

$$m = \frac{l(k-l)}{k-l-1} - 1, \quad X(\theta) = \ln(1-\theta) + \sum_{i=1}^{(l-1)/2} \cos 2\tau_i \ln(1 + 2\theta\cos\tau_i + \theta^2) + \tag{2.15}$$

$$+ 2\sum_{i=1}^{(l-1)/2} \sin 2\tau_i \operatorname{arctg}\frac{\theta + \cos\tau_i}{\sin\tau_i}, \quad \tau_i = \pi\frac{2i-1}{l}$$

At a fixed time, the solution of Eq. (1.5) must be identical to expression (2.15), and we shall therefore seek a solution of Eq. (1.5) in the form

$$x(p,t) = -\frac{l}{k}a(t)p - \frac{l}{k-l}\frac{a(t)}{b(t)^l}p^{(k-l)/k} + c(t)X(b(t)p^{1/k}) + d(t) \tag{2.16}$$

Substituting expression (2.16) into Eq. (1.5), we obtain

$$\dot{a} = 0, \quad \dot{c} = 0, \quad \dot{d} = 1/a$$

It follows from this that

$$a(t) = a = \text{const}, \quad c(t) = c = \text{const}, \quad d(t) = t/a + \alpha$$

But $c(t) = a(t)b(t)^{ml}$ (see relation (2.5)) and, consequently, $c = ab(t)^{ml}$, and, then, $b(t) = b = \text{const}$. A solution of Eq. (1.5) has been obtain which depends on three arbitrary constants $a$, $b$ and $\alpha$.

We also write the singular solution of Eq. (2.14)

$$\varphi_p = -t_\xi/\gamma \Rightarrow \varphi = \alpha p - t/(\gamma\alpha) + \beta, \quad \alpha = \text{const}, \quad \beta = \text{const}$$

So far it has been assumed that the characteristic $\xi = \text{const}$ has the form of (2.1) or (2.13). In the characteristic variables, the solution of Eq. (1.4) depends on the single variable ($t = t(\xi)$, $t_n = 0$). In the general case, we can assume that $\xi = \psi(x, p, t)$ and them, after substituting the implicitly defined function $t = f(\psi(x, p, t))$ into Eq. (1.4), we obtain ($\psi_t f_\psi \neq 1$).

$$-\psi_p^2/f_\xi + \psi_p^2\psi_t + \psi_p\psi_x^2/\gamma - p(\psi_{pp}\psi_x^2 - 2\psi_p\psi_{xp}\psi_x + \psi_p^2\psi_{xx}) = 0 \tag{2.17}$$

It is obvious that, in this case, $\psi(x, p, t) = a(t)$, where $a(t)$ is a function which is the inverse of the function $t = f(\psi)$ and $1/f_\xi = \dot{a}(t)$.

Suppose that

$$a(t) = \psi(x, p, t) = -p + \varphi(x, t)$$

Then, after substituting the derivatives of the function $\psi(x, p, t)$ into Eq. (2.17), we obtain the equation ($\psi_p = -1$)

$$-\dot{a}(t) + \varphi_t - \varphi_x^2/\gamma - (\varphi - a(t))\varphi_{xx} = 0 \tag{2.18}$$

If $t$ = const, then (2.18) is a characteristic equation. Unlike in the characteristic equations (2.3) and (2.14), there is a term $\varphi_t$ in it which, when $t$ = const, cannot be equal to zero on the characteristic. An exact solution is known [3] where, when $t$ = const, this term depends on $x$. Hence, (2.18) is the most general case of a characteristic equation.

Equation (2.18) is a second-order ODE and has a solution which depends on two arbitrary constants: $\varphi = \varphi(x, b, c)$, $b$ = const, $c$ = const. Since the solution of Eq. (1.2) on the characteristic must be identical to the solution of Eq. (2.18), if $\varphi_t$ depends on $x$, then the solution of Eq. (1.2) will have the form $p = \varphi(x, b(t), c(t)) - a(t)$, and, on substituting it into Eq. (1.2), we obtain a first-order ODE for the functions $b(t)$, $c(t)$ and $a(t)$. Consequently, the exact solution of Eq. (1.2) will have an arbitrariness of no greater than three arbitrary constants.

For example [3]

$$p(x, t) = -(x-\alpha)^2/s + \mu/s^{2/\delta}, \quad \delta = 2(\gamma+2)/\gamma, \quad s = \delta t + \beta \tag{2.19}$$

Here $\alpha$, $\beta$ and $\mu$ are arbitrary constants. It can be verified that expression (2.19) satisfies Eqs (1.2) and (2.18).

It follows from all that has been said above that any exact solution of Eq. (1.2) has an arbitrariness of no greater than three constants.

## 3. APPROXIMATE SOLUTION OF THE INITIAL-BOUNDARY-VALUE PROBLEM

We will now show how expressions for the function $p(x, t)$ with a constant arbitrariness can be used to solve initial-boundary-value problem (1.2), (1.3).

Having an exact solution which depends on three arbitrary constants, it is possible to construct a solution which depends on two arbitrary functions if two constants are given as certain functions of a third constant. Using the example of the solution (2.19), we will show that these functions can be chosen such that the initial and boundary conditions will be exactly satisfied.

According to conditions (1.3), $p(0, t) = F(t)$. We fix $t = \nu$ in expression (2.19) and require that the following equality be satisfied

$$F(\nu) = -\alpha^2 r^{-1} + \mu r^{-2/\delta}, \quad r = \delta\nu + \beta \tag{3.1}$$

By also requiring that the equality $p_t(0, \nu) = \dot{F}(\nu)$ be a satisfied for the derivative with respect to $t$ of the function (2.19), we obtain

$$\dot{F}(\nu) = \delta\alpha^2 r^{-2} - 2\mu r^{-(2/\delta+1)} \tag{3.2}$$

From relations (3.1) and (3.2), we find the values of $\alpha^2$ and $\mu$ if $r \neq 0$

$$\alpha^2 = \gamma(\dot{F}r^2 + 2Fr)/4, \quad \mu = \gamma r^{2/\delta}(\dot{F}r + \delta F)/4 \tag{3.3}$$

Satisfying conditions (3.3) means that the curve $p(0, t) = F(t)$ is the envelope of the curves formed by the sections $x = 0$ in the surface $p = p(x, t)$ from the solution (2.19).

On requiring that the conditions $\alpha_\nu = 0$ and $\mu_\nu = 0$ be satisfied for $\alpha$ and $\mu$ in the expressions (3.3), we obtain

$$L(\nu, \beta) = \ddot{F}r^2 + 2(\delta+1)\dot{F}r + 2\delta F = 0; \quad F = F(\nu), \quad r = r(\nu) \tag{3.4}$$

which is equivalent to the requirement that $p_{tt}(0, \nu) = \ddot{F}(\nu)$.

From equality (3.4) we find that $\nu = \nu(\beta)$ (if $L_\nu \neq 0$) and substitute it into expression (3.3). We obtain the required functions $\alpha = \alpha(\beta)$, $\mu = \mu(\beta)$, and substitution of these into expression (2.19) gives the one-parameter family of surfaces

$$p(x, t) = -(x - \alpha(\beta))^2 s^{-1} + \mu(\beta) s^{-2/\delta} \tag{3.5}$$

We require that the derivative $p_B$ be equal to zero

$$p_\beta = 2(x-\alpha)\alpha_\beta s^{-1} + (x-\alpha)^2 s^{-2} + \mu_\beta s^{-2/\delta} - 2\mu s^{-2/\delta-1}/\delta = 0 \tag{3.6}$$

Expressing $\beta = \beta(x, t)$ from this equality and substituting it into expression (3.5), we obtain a surface which is the envelope of the one-parameter family of surfaces (3.5).

Although the functions (3.5) are exact solutions of Eq. (1.2), the function

$$P(x, t) = p(x, t, \beta(x, t))$$

will only satisfy Eq. (1.2) approximately since

$$P_t = p_t + p_\beta \beta_t = p_t, \quad P_x = p_x + p_\beta \beta_x = p_x$$
$$P_{xx} = p_{xx} + p_{x\beta}\beta_x + (p_\beta \beta_x)_x = p_{xx} + p_{x\beta}\beta_x \neq p_{xx}$$

($p_{x\beta} \neq 0$ when $\beta = \beta(x, t)$ from the equality (3.6)).

The maximum error on substituting the approximate solution into Eq. (1.2) will be equal to $pp_{x\beta}\beta_x$, where $p = p(x', t', \beta)$, $\beta = \beta(x', t')$, and $x'$ and $t'$ are the solutions of the system of equations

$$(pp_{x\beta}\beta_x)_x = 0 \quad (pp_{x\beta}\beta_x)_t = 0 \tag{3.7}$$

In the wave front, where $P(x, t) = 0$, and $p_{x\beta}(x, t, \beta(x, t))\beta_x(x, t) \neq \infty$, Eq. (1.2) is satisfied exactly but, when $P(x, t) \neq 0$, the error can be significant.

In order to obtain a more exactly solution of the initial boundary-value problem, we shall assume that $\nu = t$ in expressions (3.1) and (3.3) and that

$$p(x, t) = -(x - \alpha(t))^2 r(t)^{-1} + \lambda(t), \quad \lambda(t) = \mu(t) r(t)^{-2/\delta} \tag{3.8}$$

where $r(t)$ is an unknown function which, when $t = \nu$, can be determined from relation (3.4) in the form

$$r(\nu) = \{-(\delta+1)\dot{F} \pm [((\delta+1)\dot{F})^2 - 2\delta F\ddot{F}]^{1/2}\}/\ddot{F}, \quad F = F(\nu) \tag{3.9}$$

if $\ddot{F} \neq 0$ and the expression in the square brackets in (3.9) is non-negative.

Table 1

| $\dot{F}$ | $F$ | $r$ | | |
|---|---|---|---|---|
| | | $\gamma > 0$ | $-2 < \gamma < 0$ | $\gamma < -2$ |
| >0 | >0 | $0 < r, r < r^* < 0$ | $r = 0$ | $r^* < r < 0$ |
| >0 | <0 | $r < 0, 0 < r^* < r$ | $r = 0$ | $0 < r < r^*$ |
| <0 | >0 | $0 < r < r_*$ | $r < r^* < 0, 0 < r_* < r$ | $r < 0, 0 < r_* < r$ |
| <0 | <0 | $r_* < r < 0$ | $r < r_* < 0, 0 < r^* < r$ | $0 < r, r < r_* < 0$ |

We shall seek the contour lines of the function $p(x, t)$. From relation (3.8), we find a function $x = x(t)$ such that $p(x(t), t) = F(\nu)$ and obtain

$$x(t) = \alpha(t) \pm [(\lambda(t) - F(\nu))r(t)]^{1/2} \tag{3.10}$$

We next substitute expression (3.8) into Eq. (1.2) and then substitute expression (3.10) into the resulting relation. We obtain an ODE for determining the function $r = r(t)$ on the contour line $p(x(t), t) = F(\nu)$ of the form

$$\dot{r} = \frac{r(g^{1/2} - \chi)(\ddot{F}r + 2\dot{F}) + \delta(\delta - 2)\chi(F - F(\nu))}{(\delta - 2)\chi(F - F(\nu)) - 2(g^{1/2} - \chi)(\dot{F}r + F)} \tag{3.11}$$
$$g = (\lambda - F(\nu)), \quad \chi = r^{-1/2}\alpha$$

We determine the initial value $r = r(\nu)$ from equality (3.9). In Eq. (3.11), where no argument is indicated in the case of the functions and derivatives, the argument is understood to be $t$.

After finding the solution of Eq. (3.11) for $t > \nu$, we substitute the value of $r(t)$ obtained into equality (3.1) and determine the value of $x = x(t)$ such that $p(x(t), t) = F(\nu)$. When $\nu = 0$, this value will correspond to the perturbation front which separates the domain of quiescence from the moving medium.

The necessary and sufficient conditions for the existence of a true perturbation front (3.10) are presented in Table 1, where

$$r_* = -2F/\dot{F}, \quad r^* = -\delta F/\dot{F} \ (F = F(t))$$

In order that the condition $x(0) = 0$ be satisfied, the relation $\operatorname{sign} \alpha = -\operatorname{sign}(gr)^{1/2}$ must also be satisfied. Hence, Eq. (1.2) will be exactly satisfied on any contour line.

*Remark.* If the value of $r(\nu)$ which satisfies relation (3.9) is such that

$$\beta(\nu) = r(\nu) - 2\delta\nu = \text{const}$$

then, in the case of the given boundary conditions, expression (3.5) is an exact solution of Eq. (1.2) since $\alpha = \text{const}$ and $\mu = \text{const}$.

*Examples.* 1. Suppose $\gamma = -1$ and $F(t) = -\operatorname{tg}(t/2)$. Substituting the functions corresponding to the specified boundary conditions into relation (3.9), we obtain that $\beta(\nu) \neq \text{const}$ and, hence, to construct the perturbation front $x = x(t)$, we solve Eq. (3.11) and substitute the resulting values of the function $r(t)$ into equality (3.10). An exact solution [5]

$$p(x, t) = \operatorname{ctg} t - \cos x / \sin t$$

is known for the boundary conditions being considered.

The perturbation front in this case, if $0 < t < \pi, x > 0$, has the form $x = t$. The approximate perturbation front differs from the exact perturbation front by less than 3% when $t \leq 0.85$ and by less than 1% when $t \leq 0.5$.

2. Suppose the boundary conditions $F(t) = (1 - e^{2t})/2$ when $\gamma = -2$. Substituting the corresponding values into equality (3.9) and choosing the minus sign in front of the root in order to satisfy condition $r \neq 0$, we obtain $\beta(\nu) = -1 = \text{const}$.

It was pointed out in the remark that, in the case of the given boundary conditions ($\beta = \text{const}$) it is possible to obtain an exact solution of the initial boundary-value problem by putting $\nu = t$. From expression (3.3), we find that

$$\alpha^2 = 1/2, \quad \mu\beta^{-2/\delta} = -e^{2t}/2$$

Substituting the values which have been found into relation (3.5), we obtain the exact solution for the given boundary conditions

$$p(x, t) = (x + 1/\sqrt{2})^2 - e^{2t}/2$$

3. We now consider the solutions of problem (1.2), (1.3) when $F(t) = at$, $a = \text{const}$. If the values of $\gamma$ and $a$ have the same signs, then, when $\beta = \infty$, we obtain the exact solution of this problem

$$p(x, t) = F(t) \mp x(\gamma\dot{F})^{1/2} \tag{3.12}$$

If $\gamma$ and $a$ have opposite signs, then the function (3.12) is complex-valued. In this case, we make use of relations (3.3) and (3.4) in order to obtain a solution. Let us assume, for example, that $\gamma = -1$, $a = c^2$. Then

$$\beta = 0, \quad \alpha = 0, \quad \mu = c^2/2$$

The exact solution in the case of the specified boundary conditions has the form

$$p(x, t) = x^2/(2t) + c^2 t$$

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